MŰEGYETEM 1782
Budapest University of Technology and Economics
Department of Applied Mechanics

PRACTICE 2 – 1 DOF UNDAMPED SYSTEM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1, a 1 DoF oscillator is shown that consists of the mass m_1, the springs of stiffness k_1 and k_2. As it can be seen one of the springs connects to a cantilever beam, which length is l, its cross section is characterized by a and b, the elastic modulus of its material refers to E. The mass of the beam is negligible. The displacement of the mass m_1 is described by the generalized coordinate x. The vibration of the system is induced by the impact between the mass m_1 and lumped mass m_2.

Data

$a = 0.006\,\mathrm{m}$	$b = 0.025\,\mathrm{m}$
$l = 0.5\,\mathrm{m}$	$E = 200\,\mathrm{GPa}$
$m_1 = 5\,\mathrm{kg}$	$m_2 = 1\,\mathrm{kg}$
$k_1 = 100\,\mathrm{N/m}$	$k_2 = 50\,\mathrm{N/m}$
$c_1 = 0\,\mathrm{m/s}$	$c_2 = 0.6\,\mathrm{m/s}$
$e = 0.5$	

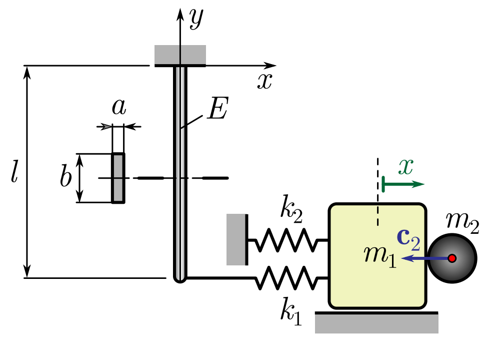

Fig. 1: The investigated system

Tasks

1. Determine the natural angular frequency of the system!
2. Calculate the maximal displacement, velocity and acceleration of the oscillation that is generated by the impact! Sketch the time histories of the displacement, velocity and acceleration!

Made by Dénes Takács

Solution

Task 1

The original mechanical system is a 1 DoF undamped oscillator, hence, it can be reduced to the system shown in Fig. 2. To obtain the equivalent stiffness k_e, first the stiffness of the beam has to be calculated. The displacement of the end point of a cantilever beam loaded by a force at its free

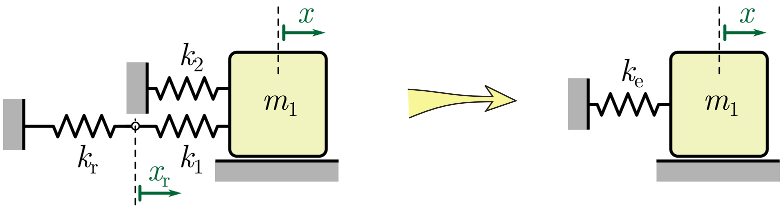

Fig. 2: The reduced model

end can be calculated by the formula:

$$f = \frac{Fl^3}{3I_zE}\,, \tag{1}$$

where $I_z = \frac{a^3b}{12} = 4.5\times 10^{-10}\,\mathrm{m}^4$ is the second moment of the cross section of the beam. The stiffness of the beam:

$$k_\mathrm{r} = \frac{F}{f} = \frac{3I_zE}{l^3} = 2160\,\mathrm{N/m}\,. \tag{2}$$

In order to obtain the equivalent stiffness in the reduced model, we can use principle of the equivalent potential energies. Namely, the potential energy in the reduced model is equal to the sum of the potential energies of the springs in the original model:

$$U = \frac{1}{2}k_\mathrm{r}x_\mathrm{r}^2 + \frac{1}{2}k_1(x-x_\mathrm{r})^2 + \frac{1}{2}k_2x^2 \quad \Leftrightarrow \quad U = \frac{1}{2}k_\mathrm{e}x^2\,, \tag{3}$$

where x_r is the displacement of the free end of the cantilever beam. This displacement can be given as a function of x since the forces in the springs k_r and k_1 are the same:

$$F_\mathrm{r} = F_1\,, \tag{4}$$

which leads to

$$k_\mathrm{r}x_\mathrm{r} = k_1(x-x_\mathrm{r})\,, \tag{5}$$

from which we obtain:

$$x_\mathrm{r} = \frac{k_1}{k_1+k_\mathrm{r}}x\,. \tag{6}$$

Substitute this into (3) and make some algebraic manipulation to have:

$$U = \frac{1}{2}\underbrace{\left(k_\mathrm{r}\left(\frac{k_1}{k_1+k_\mathrm{r}}\right)^2 + k_1\left(\frac{k_\mathrm{r}}{k_1+k_\mathrm{r}}\right)^2 + k_2\right)}_{=k_\mathrm{e}}x^2\,. \tag{7}$$

Namely, the equivalent stiffness can be given as

$$k_\mathrm{e} = \frac{k_1k_\mathrm{r}}{k_1+k_\mathrm{r}} + k_2 = 145.575\,\mathrm{N/m}\,. \tag{8}$$

The equation of motion of the reduced 1 degree-of-freedom model is

$$m_1\ddot{x} + k_\mathrm{e}x = 0 \quad \Rightarrow \quad \ddot{x} + \underbrace{\frac{k_\mathrm{e}}{m_1}}_{=\omega_\mathrm{n}^2}x = 0\,, \tag{9}$$

where the natural angular frequency is obtained as

$$\boxed{\omega_\mathrm{n} = \sqrt{\frac{k_\mathrm{e}}{m_1}} = 5.39\,\mathrm{rad/s}\,.} \tag{10}$$

The natural frequency and time period are

$$f_\mathrm{n} = \frac{\omega_\mathrm{n}}{2\pi} = 0.858\,\mathrm{Hz} \quad \text{and} \quad T_\mathrm{n} = \frac{1}{f_\mathrm{n}} = \frac{2\pi}{\omega_\mathrm{n}} = 1.166\,\mathrm{s}\,, \tag{11}$$

respectively.

Task 2

To calculate the initial condition for the impact generated oscillation, we have to investigate of the impact shown in Fig. 3. As it can be seen, the impact is centric for both bodies. Based on the normal direction of the impact, the normal velocities of the bodies before the impact are

$$c_{1\mathrm{n}} = 0 \quad \text{and} \quad c_{2\mathrm{n}} = -0.6\,\mathrm{m/s}\,, \tag{12}$$

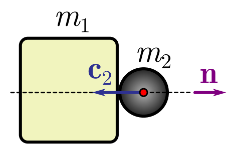

Fig. 3: The impact

respectively. The velocity of the common center of mass:

$$c_{S\mathrm{n}} = v_{S\mathrm{n}} = \frac{m_1 c_{1\mathrm{n}} + m_2 c_{2\mathrm{n}}}{m_1 + m_2} = -0.1\,\mathrm{m/s}\,. \tag{13}$$

The Maxwell diagram of the impact is shown in Fig. 4. From the Maxwell diagram, one can obtain

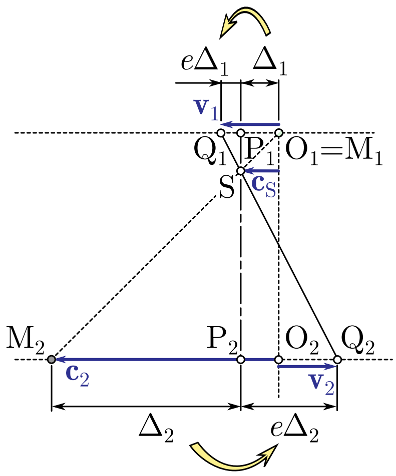

Fig. 4: The Maxwell diagram of the impact

the velocities after the impact as

$$\begin{aligned} v_{1\mathrm{n}} &= c_{S\mathrm{n}} + e\Delta_1 = c_{S\mathrm{n}} + e(c_{S\mathrm{n}} - c_{1\mathrm{n}}) = (1+e)c_{S\mathrm{n}} = -0.15\,\mathrm{m/s}\,, \\ v_{2\mathrm{n}} &= c_{S\mathrm{n}} - e\Delta_2 = c_{S\mathrm{n}} - e(c_{2\mathrm{n}} - c_{S\mathrm{n}}) = 0.15\,\mathrm{m/s}\,. \end{aligned} \tag{14}$$

Thus, the initial condition reads

$$\begin{cases} x(t=0) = 0 \\ \dot{x}(t=0) = v_0 = -0.15\,\mathrm{m/s}\,. \end{cases} \tag{15}$$

The general solution of an undamped 1 DoF oscillator can be written as

$$x(t) = C_1 \cos(\omega_\mathrm{n} t) + C_2 \sin(\omega_\mathrm{n} t)\,, \tag{16}$$

where C_1 and C_2 can be determined from the initial condition. In order to this, let us derive the general solution:

$$\dot{x}(t) = -C_1\omega_n \sin(\omega_n t) + C_2\omega_n \cos(\omega_n t)\,. \tag{17}$$

Substitute these into the initial condition:

$$\begin{cases} x(t=0) = 0 & \Rightarrow \quad C_1 \cdot 1 + C_2 \cdot 0 = 0\,, \\ \dot{x}(t=0) = v_0 & \Rightarrow \quad -C_1\omega_n \cdot 0 + C_2\omega_n \cdot 1 = v_0\,, \end{cases} \tag{18}$$

from which we obtain:

$$C_1 = 0 \quad \text{and} \quad C_2 = \frac{v_0}{\omega_n} = -0.0278\,\mathrm{m}\,. \tag{19}$$

The displacement, the velocity and the acceleration of the system are given by

$$\begin{aligned} x(t) &= -0.0278\sin(5.39t)\,\mathrm{m}\,, \\ v(t) &= \dot{x}(t) = -0.15\cos(5.39t)\,\mathrm{m/s}\,, \\ a(t) &= \ddot{x}(t) = 0.809\sin(5.39t)\,\mathrm{m/s^2}\,, \end{aligned} \tag{20}$$

respectively. The maximal displacement, velocity and acceleration are

$$\boxed{\begin{aligned} x_{\max} &= |C_2| = 0.0278\,\mathrm{m}\,, \\ v_{\max} &= |C_2\omega_n| = 0.15\,\mathrm{m/s}\,, \\ a_{\max} &= |C_2\omega_n^2| = 0.809\,\mathrm{m/s^2}\,. \end{aligned}} \tag{21}$$

The corresponding time histories are shown in Fig. 5.

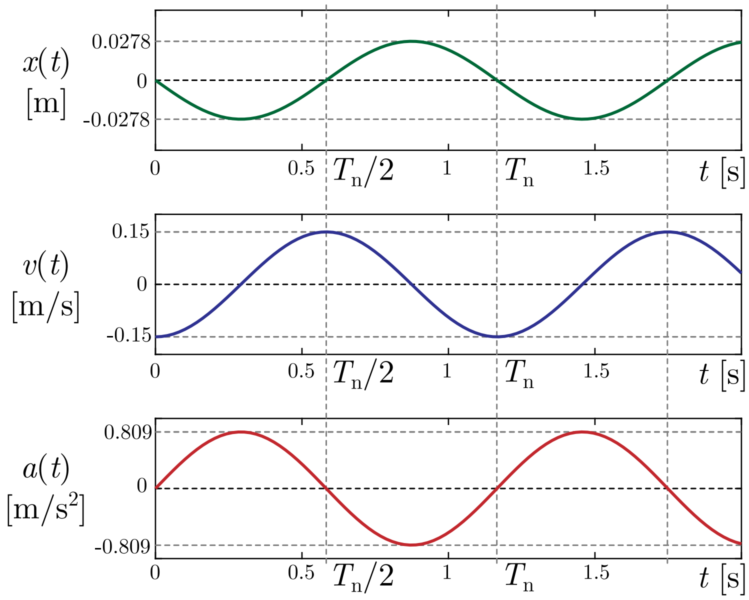

Fig. 5: The time histories of the impact generated oscillation.