Budapest University of Technology and Economics
Department of Applied Mechanics

PRACTICE 3 – 1 DOF SWINGING ARM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

The swinging arm shown in Fig. 1 consists of two mass-less rods and three lumped masses m_1, m_2 and m_3. The swinging arm can only rotate about the joint A. To describe the motion of the swinging arm, the angle φ, measured from the horizontal axis, is used as generalized coordinate. The arm is in the gravitational field. The equilibrium position is located at $\varphi = 0$, which is ensured by the preload in the spring of stiffness k.

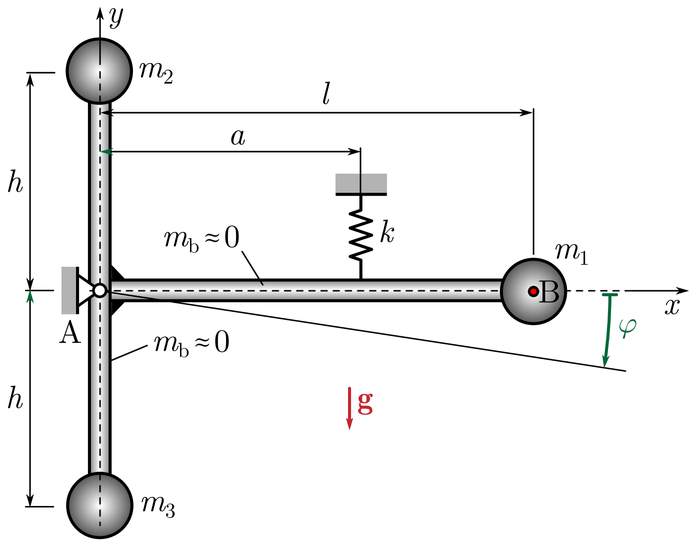

Fig. 1: The mechanical model of the swinging arm

Data

$m_1 = 2\,\mathrm{kg}$	$l = 1\,\mathrm{m}$
$m_2 = 4\,\mathrm{kg}$	$h = 0.5\,\mathrm{m}$
$m_3 = 3\,\mathrm{kg}$	$a = 0.6\,\mathrm{m}$
$k = 10^4\,\mathrm{N/m}$	$g = 9.81\,\mathrm{m/s^2}$

Tasks

1. Derive the equation of motion for small oscillation around the equilibrium!
2. Calculate the natural angular frequency, natural frequency and the period of oscillation! ($\omega_\mathrm{n} = 30.96$ rad/s, $f_\mathrm{n} = 4.93$ Hz, $T_\mathrm{n} = 0.203$ s)
3. Determine the law of motion for the initial condition given by the vertical position $y_\mathrm{B}(t=0) = -0.01\,\mathrm{m}$ and velocity $v_{\mathrm{B}y}(t=0) = -1\,\mathrm{m/s}$ of the point B! ($C_1 = 0.01$ rad, $C_2 = 0.033$ rad)
4. Calculate the maximum force in the spring for the given initial condition! ($F_{\mathrm{r,max}} = 235.56\,\mathrm{N}$)

Made by Dénes Takács

Solution

Task 1

Before the derivation of the equation of motion, it is worth to explain why the angle φ is considered as generalized coordinate in this problem. In order to have a simple form for the equation of motion, one can use two different coordinates. The generalized coordinate ψ (see Fig. 2) can be started from the position where the spring is unloaded. Of course, this cannot be done if more than one spring are in the system and they are unloaded in different positions. As another option, the generalized coordinate φ measured from the equilibrium position can be used. Since the vibrations appear around the equilibrium this latter choice is more practical.

The free body diagram of the swinging arm is shown Fig. 2, where the arm is disturbed from the equilibrium position. The relation between the possible generalized coordinates ψ and φ is also illustrated in the figure, namely:

$$\psi(t) = \psi_0 + \varphi(t)\,, \tag{1}$$

where ψ_0 is the angle (static deformation) between the unloaded spring position and the equilibrium position. The angular velocity and acceleration can be easily described by each of the coordinates, namely $\dot{\varphi} = \dot{\psi}$ and $\ddot{\varphi} = \ddot{\psi}$.

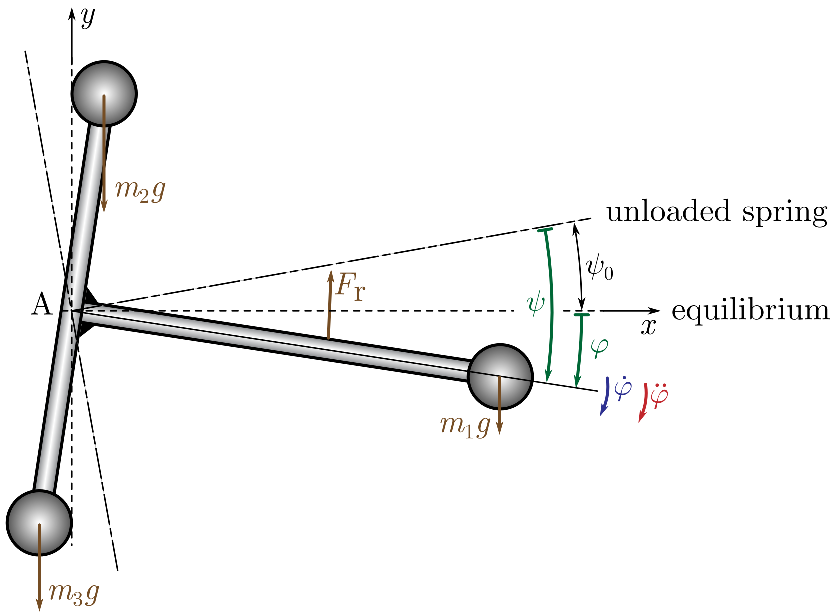

Fig. 2: *Deficient* free body diagram

Let us use φ as generalized coordinate. Based on the free body diagram, the equation of motion can be derived using the basic law of dynamics, namely:

$$\dot{\Pi}_{\mathrm{A}z} = M_{\mathrm{A}z}\,. \tag{2}$$

This translates into

$$-\,\theta_\mathrm{A}\ddot{\varphi} = -m_1 g l \cos\varphi - m_2 g h \sin\varphi + m_3 g h \sin\varphi + M_\mathrm{r}(\varphi), \tag{3}$$

where $M_\mathrm{r}(\varphi)$ represents the moment of the spring force. The exact formulation for this moment is strongly nonlinear and it also depends on the free length of the spring. However, it can be proven that the free length of the spring only appears in the fourth degree term of the Taylor expansion of the moment (see Section 2.2.2 in Csernák and Stépán 2019[1]).

[1]Csernák and Stépán: Rezgéstan, Akadémiai Kiadó, 2019,

In case of small oscillations around the equilibrium, the spring force can be approximated as

$$F_{\mathrm{r}}(\varphi) \approx F_{\mathrm{r,st}} + ka\varphi\,, \tag{4}$$

where $F_{\mathrm{r,st}}$ is the static spring force which ensures the equilibrium at $\varphi = 0$, and the second term is the so-called dynamic force $F_{\mathrm{r,dyn}} \approx ka\varphi$ generated by the vibrations. The linearized formula for the moment of the spring force can be given as

$$M_{\mathrm{r}}(\varphi) \approx M_{\mathrm{r,st}} + ka^2\varphi\,, \tag{5}$$

where $M_{\mathrm{r,st}} = F_{\mathrm{r,st}}a$ is the moment of the static spring force. Figure 3 shows the characteristic of the moment of the spring force and its linearization around the equilibrium.

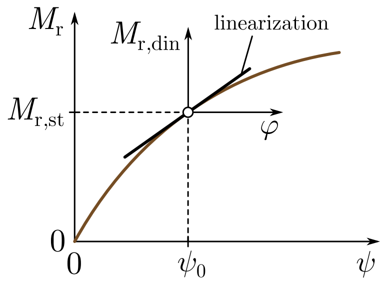

Fig. 3: The moment of the spring force and its linearization

Using Eq. (5) and considering that the linear approximations of the cosine and sine functions are $\cos\varphi \approx 1$ and $\sin\varphi \approx \varphi$, we can obtain from Eq. (3):

$$-\theta_{\mathrm{A}}\ddot{\varphi} = -m_1gl - m_2gh\varphi + m_3gh\varphi + M_{\mathrm{r,st}} + ka^2\varphi\,. \tag{6}$$

After rearranging the terms:

$$\theta_{\mathrm{A}}\ddot{\varphi} + ((m_3 - m_2)gh + ka^2)\varphi = m_1gl - M_{\mathrm{r,st}}\,. \tag{7}$$

If the equilibrium

$$\varphi \equiv 0 \quad \Rightarrow \quad \dot{\varphi} = 0\,, \quad \ddot{\varphi} = 0 \tag{8}$$

is substituted into Eq. (7), we obtain

$$0 = m_1gl - M_{\mathrm{r,st}}\,. \tag{9}$$

Namely, the right hand side of Eq. (7) is zero, and the moment of the static spring force:

$$M_{\mathrm{r,st}} = m_1gl\,. \tag{10}$$

Then, the static spring force:

$$F_{\mathrm{r,st}} = \frac{M_{\mathrm{r,st}}}{a} = m_1g\frac{l}{a}\,. \tag{11}$$

As a consequence of the calculation above, we can conclude that if the general coordinate is measured from the equilibrium position, the moment of the static spring force can be neglected in the equation of motion together with the moment of the gravitational force that generates the preload in the spring in the equilibrium. In the future, we will use this assumption in many problems to simplify the derivation of the equation of motion for small vibrations.

Dividing Eq. (7) by θ_{A} leads to the general form of the equation of motion:

$$\boxed{\ddot{\varphi} + \frac{(m_3 - m_2)gh + ka^2}{\theta_{\mathrm{A}}}\varphi = 0\,.} \tag{12}$$

Task 2

The coefficient of φ in Eq. (12) is equal to ω_n^2. The mass moment of inertia of the swinging arm at the point A with respect to the rotational axis can be calculated as

$$\theta_A = m_1 l^2 + m_2 h^2 + m_3 h^2 = 3.75\,\mathrm{kgm}^2\,, \tag{13}$$

and the natural angular frequency:

$$\boxed{\omega_\mathrm{n} = \sqrt{\frac{(m_3 - m_2)gh + ka^2}{\theta_\mathrm{A}}} = 30.96\,\mathrm{rad/s}\,.} \tag{14}$$

The natural frequency

$$\boxed{f_\mathrm{n} = \frac{\omega_\mathrm{n}}{2\pi} = 4.93\,\mathrm{Hz}} \tag{15}$$

and the period of oscillation

$$\boxed{T_\mathrm{n} = \frac{2\pi}{\omega_\mathrm{n}} = 0.203\,\mathrm{s}\,.} \tag{16}$$

Task 3

The initial conditions are given based on the position and velocity of the point B. They have to be rephrased with respect ot the generalized coordinate φ as

$$\begin{cases} \varphi(t=0) = \varphi_0 \approx -\frac{y_\mathrm{B}(t=0)}{l} = 0.01\,\mathrm{rad}\,, \\ \dot{\varphi}(t=0) = \omega_0 \approx -\frac{v_{\mathrm{B}y}(t=0)}{l} = 1\,\mathrm{rad/s}\,. \end{cases} \tag{17}$$

The general solution of an undamped 1 DoF oscillator can be written as

$$\varphi(t) = C_1 \cos(\omega_\mathrm{n} t) + C_2 \sin(\omega_\mathrm{n} t)\,, \tag{18}$$

where C_1 and C_2 can be determined from the initial conditions. The derivative of the general solution with respect to time:

$$\dot{\varphi}(t) = -C_1\omega_\mathrm{n} \sin(\omega_\mathrm{n} t) + C_2\omega_\mathrm{n} \cos(\omega_\mathrm{n} t)\,. \tag{19}$$

Let us substitute these into the initial conditions:

$$\begin{cases} \varphi(t=0) = \varphi_0 & \Rightarrow \quad C_1 \cdot 1 + C_2 \cdot 0 = \varphi_0\,, \\ \dot{\varphi}(t=0) = \omega_0 & \Rightarrow \quad -C_1\omega_\mathrm{n} \cdot 0 + C_2\omega_\mathrm{n} \cdot 1 = \omega_0\,, \end{cases} \tag{20}$$

from which we obtain:

$$C_1 = \varphi_0 = 0.01\,\mathrm{rad} \quad \text{and} \quad C_2 = \frac{\omega_0}{\omega_\mathrm{n}} = 0.033\,\mathrm{rad}\,. \tag{21}$$

Thus, the law of motion reads

$$\boxed{\varphi(t) = 0.01\cos(30.96t) + 0.033\sin(30.96t)\ \mathrm{rad}\,.} \tag{22}$$

Task 4

According to Eq. (4), one can calculate the force in the spring as

$$F_{\mathrm{r}}(t) = F_{\mathrm{r,st}} + ka\varphi(t). \tag{23}$$

The static spring force is given in Eq. (11). The maximal spring force emerges when φ is maximal:

$$F_{\mathrm{r,max}} = F_{\mathrm{r,st}} + ka\varphi_{\mathrm{max}}\,. \tag{24}$$

In order to determine the maximal value of φ, one can use the condensed form of the general solution:

$$\varphi(t) = \Phi\cos(\omega_{\mathrm{n}}t + \delta)\,, \tag{25}$$

where the amplitude Φ and the phase-shift δ can be calculated by the expanding the condensed form:

$$\Phi\cos(\omega_{\mathrm{n}}t + \delta) = \Phi\cos(\omega_{\mathrm{n}}t)\cos\delta - \Phi\sin(\omega_{\mathrm{n}}t)\sin\delta\,. \tag{26}$$

Comparing the coefficients of $\cos(\omega_{\mathrm{n}}t)$ and $\sin(\omega_{\mathrm{n}}t)$ with Eq. (18), we obtain:

$$\begin{array}{lrcl} \cos(\omega_{\mathrm{n}}t): & C_1 & = & \Phi\cos\delta\,, \\ \sin(\omega_{\mathrm{n}}t): & C_2 & = & -\Phi\sin\delta\,. \end{array} \tag{27}$$

Calculate the square of each equations and sum them to determine the amplitude:

$$\Phi = \sqrt{C_1^2 + C_2^2} = 0.0338\,\mathrm{rad}\,, \tag{28}$$

or divide the equations with each other to obtain the phase-shift:

$$\delta = -\arctan\left(\frac{C_2}{C_1}\right) = -1.277\,\mathrm{rad}\,. \tag{29}$$

The maximal deflection angle can be calculated easily, i.e. $\varphi_{\mathrm{max}} = \Phi$. Thus, the maximal spring force:

$$\boxed{F_{\mathrm{r,max}} = m_1 g\frac{l}{a} + ka\Phi = 235.56\,\mathrm{N}\,.} \tag{30}$$