Budapest University of Technology and Economics
Department of Applied Mechanics

PRACTICE 6 – 1 DOF FORCED DAMPED SWINGING ARM

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1(a) and (b), a swinging arm is shown that is modelled by a rod with length l and mass m. The swinging arm can only rotate about joint A. Two cases are distinguished: harmonic force excitation (see panel (a)) and harmonic displacement excitation that is applied through a spring with stiffness k_0 (see panel (b)). In both cases, the rod is connected to the environment through a spring with stiffness k (in case (a)) or k_1 (in case (b)) and a damper with damping factor c. To describe the motion of the corresponding 1 DoF swinging arm, the angle φ measured from the horizontal axis is used as generalised coordinate. The structure lies on the horizontal plane and its equilibrium position is located at $\varphi = 0$. In this equilibrium position, the springs with stiffness k (case (a)) and k_1 (case (b)) are unloaded.

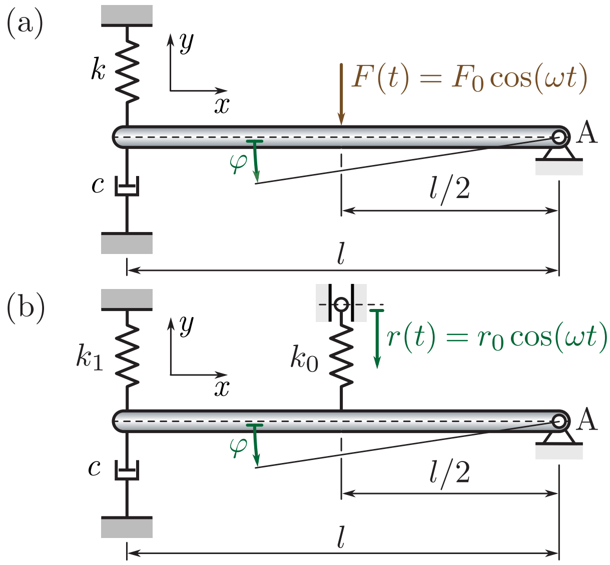

Fig. 1: Mechanical model of a swinging arm. (a) Force excitation. (b) Displacement excitation.

Data

$m = 3\,\mathrm{kg}$	$l = 1\,\mathrm{m}$	$\omega = 30\,\mathrm{rad/s}$
$k = 400\,\mathrm{N/m}$	$F_0 = 10\,\mathrm{N}$	$c = 28\,\mathrm{Ns/m}$
$k_0 = 1000\,\mathrm{N/m}$	$r_0 = 0.01\,\mathrm{m}$	$k_1 = 150\,\mathrm{N/m}$

Tasks

1. Derive the equation of motion for both models and calculate the natural angular frequencies of the undamped and damped system, the damping ratio and the static deformation! (For both cases: $\omega_\mathrm{n} = 20$ rad/s, $\omega_\mathrm{d} = 14.28$ rad/s, $\zeta = 0.7$ [1], $f_0 = 0.0125$ rad)
2. Draw the resonance curve and the phase diagram of the systems!
3. Determine the law of motion $\varphi(t)$ if the initial conditions are $\varphi(t = 0) = \varphi_0 = 0.015$ rad and $\dot{\varphi}(t = 0) = 0$ rad/s!

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Solution

Task 1

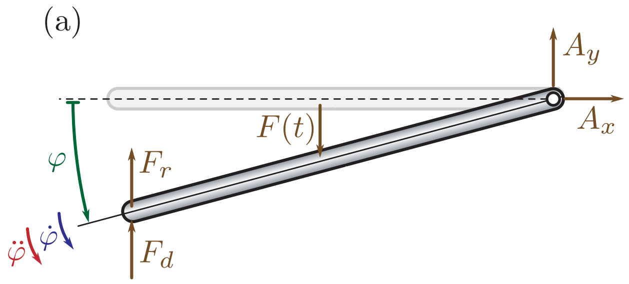

Fig. 2: *Deficient* free body diagrams

Figure 2 shows both the free body diagrams of the corresponding swinging arms in a disturbed position. To construct the governing equations of motion the basic theorem of dynamics has to be applied

$$\dot{\mathbf{\Pi}}_{\mathrm{A}} = \mathbf{M}_{\mathrm{A}}\,, \tag{1}$$

where A is a fixed point, that is, $\mathbf{v}_{\mathrm{A}} \equiv \mathbf{0}$. To obtain the equation of motion for small vibrations around the equilibrium, we use the linearised expressions (see the previous examples of the course) to calculate the moments of the forces acting on the swinging arm. Thus, for cases (a) and (b) Eq. (1) translates into

$$\theta_{\mathrm{A}}\ddot{\varphi}(t) = F(t)\frac{l}{2} - F_r l - F_d l\,, \tag{2a}$$

$$\theta_{\mathrm{A}}\ddot{\varphi}(t) = -F_{r_0}(t)\frac{l}{2} - F_{r_1} l - F_d l\,, \tag{2b}$$

where the spring and damping forces are

$$F_r \cong kl\varphi \quad \text{and} \quad F_d \cong cl\dot{\varphi}\,. \tag{3a}$$

$$F_{r_1} \cong k_1 l\varphi \quad \text{and} \quad F_d \cong cl\dot{\varphi}\,. \tag{3b}$$

The excitation takes the form

$$F(t) = F_0\cos(\omega t)\,. \tag{4a}$$

$$F_{r_0}(t) \cong k_0\left(\frac{l}{2}\varphi - r_0\cos(\omega t)\right)\,. \tag{4b}$$

The mass moment of inertia of the swinging arm at the point A with respect to the rotational axis z can be calculated by means of the parallel-axis theorem (Steiner's theorem), that is

$$\theta_{\mathrm{A}} = \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3}ml^2\,. \tag{5}$$

Accordingly, the substitution of Eqs. (3a) and (4a) into (2a), and Eqs. (3b) and (4b) into (2b), respectively, gives the governing equations of motion

$$\frac{ml^2}{3}\ddot{\varphi} + cl^2\dot{\varphi} + kl^2\varphi = F_0\frac{l}{2}\cos(\omega t)\,. \tag{6a}$$

$$\frac{ml^2}{3}\ddot{\varphi} + cl^2\dot{\varphi} + \left(k_1 + \frac{k_0}{4}\right)l^2\varphi = k_0 r_0\frac{l}{2}\cos(\omega t)\,. \tag{6b}$$

Dividing the equations by $ml^2/3$ results in the general forms

$$\ddot{\varphi} + \underbrace{\frac{3c}{m}}_{=2\zeta\omega_{\mathrm{n}}}\dot{\varphi} + \underbrace{\frac{3k}{m}}_{=\omega_{\mathrm{n}}^2}\varphi = \underbrace{\frac{3F_0}{2ml}}_{=f_0\omega_n^2}\cos(\omega t)\,. \tag{7a}$$

$$\ddot{\varphi} + \underbrace{\frac{3c}{m}}_{=2\zeta\omega_{\mathrm{n}}}\dot{\varphi} + \underbrace{\frac{3k_1 + \frac{3}{4}k_0}{m}}_{=\omega_{\mathrm{n}}^2}\varphi = \underbrace{\frac{3k_0 r_0}{2ml}}_{=f_0\omega_n^2}\cos(\omega t)\,. \tag{7b}$$

Substituting the numerical parameters given above leads to the natural angular frequency of the undamped system

$$\boxed{\omega_\mathrm{n} = \sqrt{\frac{3k}{m}} = 20\,\mathrm{rad/s}} \tag{8a}$$

$$\boxed{\omega_\mathrm{n} = \sqrt{\frac{3k_1 + \frac{3}{4}k_0}{m}} = 20\,\mathrm{rad/s}} \tag{8b}$$

and to damping ratio (same for both cases)

$$\boxed{\zeta = \frac{1}{2\omega_\mathrm{n}}\frac{3c}{m} = 0.7\,(= 70\%)\,.} \tag{9}$$

The natural angular frequency of the damped system is (same for both cases)

$$\boxed{\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2} = 14.28\,\mathrm{rad/s}\,,} \tag{10}$$

correspondingly, the period of oscillation

$$T_\mathrm{d} = \frac{2\pi}{\omega_\mathrm{d}} = 0.4399\,\mathrm{s} \tag{11}$$

and the frequency of the oscillation

$$f_\mathrm{d} = \frac{\omega_\mathrm{d}}{2\pi} = 2.2727\,\mathrm{Hz}\,. \tag{12}$$

The static deformation f_0 of the corresponding system can be calculated by means of the right hand sides of Eqs. (7a) and (7b):

$$f_0 = \frac{1}{\omega_\mathrm{n}^2}\frac{3F_0}{2ml} = \frac{F_0}{2kl} = 0.0125\,\mathrm{rad}\,. \tag{13a}$$

$$f_0 = \frac{1}{\omega_\mathrm{n}^2}\frac{3k_0r_0}{2ml} = 0.0125\,\mathrm{rad}\,. \tag{13b}$$

Consequently, the governing equations of motion of the different configurations are the same because of the specific parameter setups of the different models.

Task 2

The solution $\varphi(t)$ of the equation of motion can be decomposed as

$$\varphi(t) = \varphi_\mathrm{h}(t) + \varphi_\mathrm{p}(t)\,, \tag{14}$$

where φ_h is the homogeneous solution and φ_p is the particular solution (also called as stationary motion) that is assumes the form

$$\varphi_\mathrm{p}(t) = \Phi\cos(\omega t - \vartheta)\,, \tag{15}$$

where Φ is the amplitude of the stationary motion and ϑ is the phase angle. The resonance curve is given by the dimensionless function

$$N = \frac{1}{\sqrt{(1-\lambda^2)^2 + 4\zeta^2\lambda^2}}\,, \tag{16}$$

where $\lambda = \omega/\omega_\mathrm{n} = 1.5$ [1] is the frequency ratio. By substituting the numerical value of λ into Eq. (16), it gives $N = 0.4092$ [1]. The amplitude of the stationary motion can be obtained by

$$\Phi = Nf_0 = 0.005115\,\mathrm{rad} \tag{17}$$

and the phase angle is given by

$$\vartheta = \arctan\frac{2\zeta\lambda}{1-\lambda^2} = -1.034 + j\pi\ \mathrm{rad}\,,\quad j \in \mathbb{N}\,. \tag{18}$$

Since the phase angle is defined on the set $\vartheta \in [0,\pi]$, j is chosen to be 1 in Eq. (18) yielding $\vartheta = 2.108$ rad.

The resonance curve and the phase diagram with the corresponding calculated system parameters can be seen in Fig. 3.

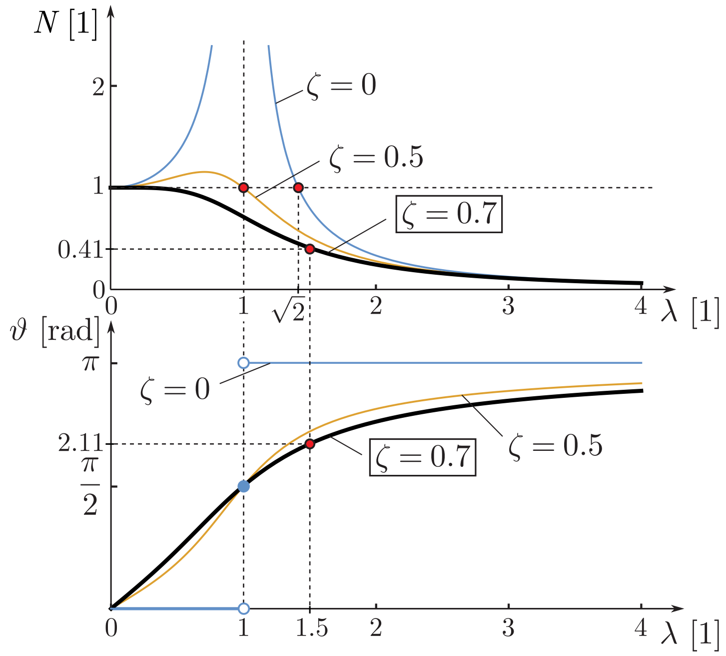

Fig. 3: Resonance curve and phase diagram

Task 3

Based on the initial conditions, the law of motion $\varphi(t)$ can be calculated. The system is modelled by a second-order-differential equation. The general solution is the superposition of the transient motion and the stationary motion (see Eq. (14)), which takes the form

$$\varphi(t) = \mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(C_1\cos(\omega_\mathrm{d}t) + C_2\sin(\omega_\mathrm{d}t)\right) + \Phi\cos(\omega t - \vartheta)\,, \tag{19}$$

and its derivetive with respect to time is

$$\begin{aligned}\dot{\varphi}(t) = &-\zeta\omega_\mathrm{n}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(C_1\cos(\omega_\mathrm{d}t) + C_2\sin(\omega_\mathrm{d}t)\right)\\ &+\omega_\mathrm{d}\mathrm{e}^{-\zeta\omega_\mathrm{n}t}\left(-C_1\sin(\omega_\mathrm{d}t) + C_2\cos(\omega_\mathrm{d}t)\right) - \Phi\omega\sin(\omega t - \vartheta)\,.\end{aligned} \tag{20}$$

The coefficients C_1 and C_2 can be determined from the initial conditions, which are

$$\begin{cases}\varphi(t=0) = \varphi_0 & \Rightarrow \quad C_1\cdot 1 + C_2\cdot 0 + \Phi\cos(-\vartheta) = \varphi_0\,,\\ \dot{\varphi}(t=0) = 0 & \Rightarrow \quad -\zeta\omega_\mathrm{n}\left(C_1\cdot 1 + C_2\cdot 0\right) + \omega_\mathrm{d}\left(-C_1\cdot 0 + C_2\cdot 1\right) - \Phi\omega\sin(-\vartheta) = 0\,.\end{cases} \tag{21}$$

This leads to

$$C_1 = \varphi_0 - \Phi\cos(-\vartheta) = 0.0176\,\mathrm{rad} \quad \text{and} \quad \mathrm{C}_2 = \frac{\zeta\omega_\mathrm{n}}{\omega_\mathrm{d}}\mathrm{C}_1 + \frac{\Phi\omega}{\omega_\mathrm{d}}\sin(-\vartheta) = 0.00803\,\mathrm{rad}\,. \tag{22}$$

By substituting these coefficients and the previously calculated system parameters into Eq. (19), the law of motion reads

$$\varphi(t) = \mathrm{e}^{-14t}\left(0.0176\cos(14.28t) + 0.00803\sin(14.28t)\right) + 0.005115\cos(30t - 2.108)\,, \tag{23}$$

which is illustrated in Fig. 4.

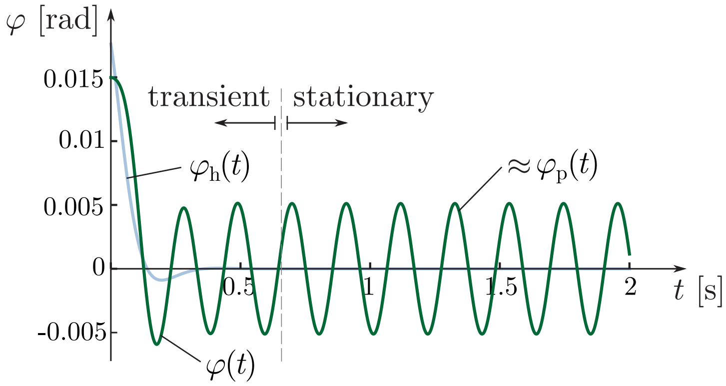

Fig. 4: Time history of generalised coordinate