MŰEGYETEM 1782

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PRACTICE 7 – 1 DOF QUARTER CAR MODEL

VIBRATIONS

– BMEGEMMBXM4 –

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Example

In Fig. 1 a quarter car model is shown that can be used to investigate the vertical dynamics of a vehicle. In practice, the stiffness and damping parameters of the suspension system are usually tuned by means of the model in order to achieve the desired road comfort. The mass of the quarter car is denoted by m, while the equivalent stiffness and damping parameters of the suspension and wheel are denoted by k and c, respectively. The road, on which the car is driven with constant longitudinal speed v, is considered to have a sinusoidal shape described by $r(t) = R\sin(\omega t)$, where R is the amplitude of the road disturbances and ω is the excitation frequency characterized by the speed v and the wavelength L. To describe the motion of the quarter car model the vertical displacement $y(t)$ is used as general coordinate. The effect of gravity is neglected and it is assumed that the wheel never lose the contact with the road.

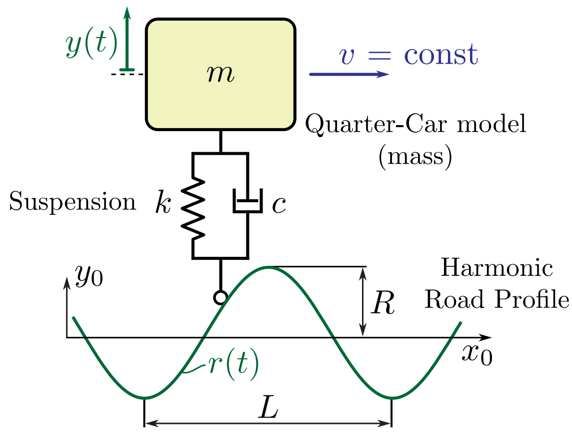

Fig. 1: Mechanical model of the quarter-car model.

Data

$m = 300\,\mathrm{kg}$	$k = 2\cdot 10^5\,\mathrm{N/m}$	$c = 9300\,\mathrm{Ns/m}$	$v = 36\,\mathrm{km/h} = 10\,\mathrm{m/s}$
$R = 0.04\,\mathrm{m}$	$L = 1.2\,\mathrm{m}$		
$r(t) = R\sin(\omega t)$			

Tasks

1. Derive the equation of motion for the quarter car model!
2. Calculate the natural angular frequency of the damped an the undamped system, the damping ratio, the frequency ratio and the static deformation! ($\omega_{\mathrm{n}} = 25.82\,\mathrm{rad/s}$, $\omega_{\mathrm{d}} = 14.28\,\mathrm{rad/s}$, $\zeta = 0.6$, $\lambda = 2.03$, $f_0 = 0.105\,\mathrm{m}$)
3. Determine the stationary motion $y_{\mathrm{p}}(t) = Y\sin(\omega t + \delta - \vartheta)$of the vehicle! ($Y = 0.0266\,\mathrm{m}$, $\delta = 1.181\,\mathrm{rad}$, $\vartheta = 2.478\,\mathrm{rad}$)!
4. Determine the maximum damping force $F_{\mathrm{d,max}}$ during the stationary motion! ($F_{\mathrm{d,max}} = 20257\,\mathrm{N}$)

Made by Henrik Sykora

Solution

Task 1

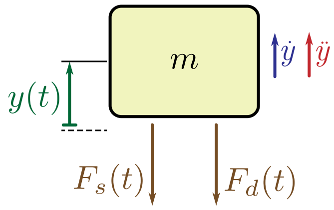

Fig. 2: Free body diagram of the quarter-car model

Figure 2 shows the free body diagram of the quarter car model in a disturbed position. To construct the governing equation of motion the basic theorem of dynamics has to be applied

$$\dot{\mathbf{I}} = \mathbf{F}. \tag{1}$$

Since the system's motion is investigated only in the vertical direction, Eq. (1) translates to

$$m\ddot{y} = -F_\mathrm{s}(t) - F_\mathrm{d}(t), \tag{2}$$

where F_s and F_d are the spring and damping forces, respectively:

$$F_\mathrm{s}(t) = k(y - r(t)), \tag{3}$$
$$F_\mathrm{d}(t) = c(\dot{y} - \dot{r}(t)). \tag{4}$$

Note that the road excitation is embedded in both the spring and the damping forces. The road excitation and its derivative can be written as:

$$r(t) = R\sin(\omega t), \tag{5}$$
$$\dot{r}(t) = R\omega\cos(\omega t), \tag{6}$$

respectively. The excitation frequency can be determined by using the wavelength of the road and the longitudinal speed of the vehicle:

$$T_\mathrm{g} = \frac{L}{v} \quad \Longrightarrow \quad \frac{2\pi}{\omega} = \frac{L}{v} \quad \Longrightarrow \quad \omega = 2\pi\frac{v}{L} = 52.36\,\mathrm{rad/s}. \tag{7}$$

Expanding the spring and damping forces described in Eqs. (3) and (4) leads to

$$m\ddot{y} + c\dot{y} + ky = kR\sin(\omega t) + cR\omega\cos(\omega t). \tag{8}$$

The excitation on the right hand side, which are related to the spring and damping forces, can be reduced to a condensed form of $F_0\sin(\omega t + \delta)$, where δ is a phase-shift compared to the original road excitation, namely:

$$kR\sin(\omega t) + cR\omega\cos(\omega t) = F_0\sin(\omega t + \delta) \tag{9}$$

The force amplitude F_0 and the phase-shift δ can be determined, by expanding the original condensed form of the force excitation:

$$F_0\sin(\omega t + \delta) = F_0\sin(\omega t)\cos(\delta) + F_0\cos(\omega t)\sin(\delta), \tag{10}$$

and then comparing the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$:

$$kR\sin(\omega t) + cR\omega\cos(\omega t) \overset{!}{=} F_0\sin(\omega t)\cos(\delta) + F_0\cos(\omega t)\sin(\delta), \tag{11}$$

$$\begin{aligned} \sin(\omega t): \qquad kR &= F_0 \cos(\delta), \\ \cos(\omega t): \qquad cR\omega &= F_0 \sin(\delta). \end{aligned} \tag{12}$$

Using the coefficients in Eq. (12), the force amplitude F_0 and the phase angle δ can be calculated as

$$F_0 = \sqrt{k^2 R^2 + c^2 R^2 \omega^2} = 21056.8\,\mathrm{N}, \tag{13}$$

$$\delta = \arctan\left(\frac{c\,\omega}{k}\right) = 1.182\,\mathrm{rad}\,. \tag{14}$$

Let us use Eq. (9) in Eq. (8) and divide it by m in order to obtain the equation of motion of the quarter-car model in the general form:

$$\boxed{\ddot{y} + \underbrace{\frac{c}{m}}_{2\zeta\omega_\mathrm{n}} \dot{y} + \underbrace{\frac{k}{m}}_{\omega_\mathrm{n}^2} y = \underbrace{\frac{R}{m}\sqrt{k^2 + c^2\omega^2}}_{f_0\omega_\mathrm{n}^2} \sin\left(\omega t + \delta\right)}\,. \tag{15}$$

Task 2

Substituting the numerical parameters given in the example description leads to

$$\boxed{\omega_\mathrm{n} = \sqrt{\frac{k}{m}} = 25.82\,\mathrm{rad/s},} \quad \Longrightarrow \quad \boxed{\lambda = \frac{\omega}{\omega_\mathrm{n}} = 2.03,} \tag{16}$$

$$\boxed{\zeta = \frac{1}{2\omega_\mathrm{n}} \frac{c}{m} = 0.6 = 60\%,} \quad \Longrightarrow \quad \boxed{\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2} = 20.65\,\mathrm{rad/s},} \tag{17}$$

$$\boxed{f_0 = \frac{1}{\omega_\mathrm{n}^2} \frac{R}{m} \sqrt{k^2 + c^2\omega^2} = 0.105\,\mathrm{m},} \tag{18}$$

where ω_n is the undamped, while ω_d is the damped natural angular frequency, λ is the frequency ratio, ζ is the damping ratio and f_0 is the static deformation.

Task 3

The stationary motion of the system can be described by the particular solution $y_\mathrm{p}(t)$ of the differential equation (15):

$$y_\mathrm{p}(t) = Y \sin\left(\omega t + \delta - \vartheta\right), \tag{19}$$

where Y is the amplitude of stationary vibration and ϑ is the phase shift compared to the force excitation. Note that $\delta - \vartheta$ is the phase shift compared to the road excitation. To determine the vibration amplitude Y the magnification N is needed:

$$N = \frac{1}{\sqrt{\left(1-\lambda^2\right)^2 + 4\zeta^2\lambda^2}} = 0.253\,. \tag{20}$$

With this the amplitude Y is

$$\boxed{Y = N f_0 = 0.0266\,\mathrm{m}\,.} \tag{21}$$

The phase angle ϑ is:

$$\boxed{\vartheta = \arctan\left(\frac{2\zeta\lambda}{1-\lambda^2}\right) = -0.664 + \pi = 2.478\,\mathrm{rad}\,,} \tag{22}$$

leading to the stationary motion

$$\boxed{y_\mathrm{p}(t) = 0.0266 \sin\left(52.36t - 1.297\right)\ \mathrm{m}\,.} \tag{23}$$

Task 4

The damping force during the stationary motion can be written as

$$F_{\mathrm{d}}(t) = c\,(\dot{y}_{\mathrm{p}}(t) - \dot{r}(t)) = c\,\omega\,\underbrace{(Y\cos(\omega t + \delta - \vartheta) - R\cos(\omega t))}_{=Y^*\cos(\omega t+\delta^*)}. \tag{24}$$

To calculate the force amplitude $F_{\mathrm{d,max}}$, the amplitude of the damping force has to be calculated. Let us introduce the condensed form: $F_{\mathrm{d}}(t) = c\,\omega\,Y^*\cos(\omega t + \delta^*)$. The term in the parenthesis of Eq. (24) can be rewritten as

$$\begin{aligned} Y\cos(\omega t)\cos(\delta - \vartheta) - Y\sin(\omega t)\sin(\delta - \vartheta) - R\cos(\omega t) = \\ Y^*\cos(\omega t)\cos(\delta^*) - Y^*\sin(\omega t)\sin(\delta^*). \end{aligned} \tag{25}$$

and the coefficient of the trigonometric function can be separated:

$$\begin{array}{lrcl} \sin(\omega t): & -Y\sin(\delta - \vartheta) & = & -Y^*\sin(\delta^*), \\ \cos(\omega t): & Y\cos(\delta - \vartheta) - R & = & Y^*\cos(\delta^*), \end{array} \tag{26}$$

from where Y^* follows as:

$$Y^* = \sqrt{Y^2 - 2RY\cos(\delta - \vartheta) + R^2} = 0.0416\,\mathrm{m}. \tag{27}$$

The amplitude of the damping force $F_d(t)$ is:

$$\boxed{F_{\mathrm{d,max}} = c\,\omega\,Y^* = 20257\,\mathrm{N}.} \tag{28}$$

Note that similarly the amplitude of the spring force $F_{\mathrm{s}}(t)$ can be determined, leading to:

$$\boxed{F_{s,\mathrm{max}} = k\,Y^* = 8320\,\mathrm{N}.} \tag{29}$$